On the rank preserving property of linear sections and its applications in tensors

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Overview

Rank decomposition

Let $V_1, \ldots, V_n$ be vector spaces over $\mathbb{K}$, and $T \in V_1 \otimes \cdots \otimes V_n$.

Definition

The *rank*, denoted by $\mathrm{rk}(T)$, of a tensor T is the minimum integer r such that

$$T = \sum_{i=1}^{r} u_{i,1} \otimes \cdots \otimes u_{i,n}$$

where each vector $u_{i,j} \in V_j$.

Such a decomposition $T = \sum_{i=1}^{r} u_{i,1} \otimes \cdots \otimes u_{i,n}$ is called a *rank-r decomposition*.

Symmetric rank decomposition

Let V be a vector space over $\mathbb{K}$, and $T \in S^d V$ be a symmetric tensor.

Definition

The *symmetric rank*, denoted by $\mathrm{rk}_S(T)$, of T is the minimum integer r such that

$$T = \sum_{i=1}^{r} \lambda_i u_i^d$$

where each vector $u_i \in V$ and each $\lambda_i \in \mathbb{K}$.

Such a decomposition $T = \sum_{i=1}^{r} u_i^d$ is called a *symmetric rank-r decomposition*, or a *Waring decomposition*.

Vandermonde rank decomposition

Let V be an $(n+1)$-dimensional vector space. Fix a basis $\{e_1, \ldots, e_{n+1}\}$ for V. A symmetric tensor

$$H := \sum_{1 \leq i_1, \ldots, i_d \leq n+1} H_{i_1 \ldots i_d} e_{i_1} \cdots e_{i_d} \in S^d V$$

is called *Hankel* if there is a vector $h := (h_0, \ldots, h_{nd})$ such that

$$H_{i_1 \ldots i_d} = h_{i_1 + \cdots + i_d - d}.$$

H is said to have a *Vandermonde rank decomposition* if, after identifying V with $S^n W$ for some 2-dim vector space W, H has the form

$$H = \sum_{i=1}^{r} \lambda_i (w_i^{\otimes n})^{\otimes d}, \tag{1}$$

where $w_1, \ldots, w_r \in W$. The minimum r is called the *Vandermonde rank* of H.

A symmetric tensor H is Hankel if and only if H has a Vandermonde rank decomposition.

Border rank

The set of tensors with rank $\leq r$ is not necessarily closed when $r > 1$.

Definition

The *border rank*, denoted by $\mathrm{brk}(T)$, of a tensor T is the minimum integer r such that T is a limit of rank-r tensors.

Definition

The *symmetric border rank*, denoted by $\mathrm{brk}_S(T)$, of a symmetric tensor T is the minimum integer r such that T is a limit of symmetric rank-r tensors.

Conjectures on rank decompositions

Conjecture (Comon)

Given any symmetric tensor $T \in S^d V$,

$$\mathrm{rk}_S(T) = \mathrm{rk}(T).$$

Conjecture (Strassen)

Given vector spaces $V_1, \ldots, V_n, W_1, \ldots, W_n$ such that $V_i \cap W_i = \{0\}$ for each i, and tensors $A \in V_1 \otimes \cdots \otimes V_n$ and $B \in W_1 \otimes \cdots \otimes W_n$. Then

$$\mathrm{rk}(A + B) = \mathrm{rk}(A) + \mathrm{rk}(B),$$

where $A + B \in (V_1 \oplus W_1) \otimes \cdots \otimes (V_n \oplus W_n)$.

Conjectures on rank decompositions continued

Conjecture (symmetric version of Strassen's conjecture)

Given vector spaces V and W such that $V \cap W = \{0\}$, and tensors $A \in S^d V$ and $B \in S^d W$. Then

$$\mathrm{rk}_S(A + B) = \mathrm{rk}_S(A) + \mathrm{rk}_S(B),$$

where $A + B \in S^d(V \oplus W)$.

Conjecture (Nie – Ye)

For a general Vandermonde rank-r Hankel tensor, its symmetric rank and rank are also r.

Overview

X-rank decomposition

Let $X \subset \mathbb{P}V$ be a nondegenerate projective variety.

- nondegenerate (X is not contained in a hyperplane) $\implies$ for any $v \in V$, $v = x_1 + \cdots + x_m$ for some $x_1, \ldots, x_m \in \widehat{X}$.
- projective $\implies$ $v = x_1 + \cdots + x_m$ instead of $v = c_1 x_1 + \cdots + c_r x_m$ for some coefficients $c_1, \ldots, c_m$.

Definition (Zak)

For $v \in V$, the *X-rank* of v, denoted by $\mathrm{rk}_X(v)$, is the minimum integer r such that

$$v = x_1 + \cdots + x_r,$$

where $x_1, \ldots, x_r \in \widehat{X}$.

Border and Generic X-rank

Definition

For $v \in V$, the *X-border-rank* of v, denoted by $\mathrm{brk}_X(v)$, is the minimum integer r such that v is a limit of X-rank-r points.

Definition

Over $\mathbb{C}$, there is a unique X-rank r such that the set of X-rank-r points contains a Zariski open subset of V, which is called the *generic* rank.

Examples

The Segre variety is defined to be the image of

$$\begin{aligned} \mathrm{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n &\to \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \\ ([v_1], \ldots, [v_n]) &\mapsto [v_1 \otimes \cdots \otimes v_n]. \end{aligned}$$

The Veronese variety is defined to be the image of

$$\nu_d : \mathbb{P}V \to \mathbb{P}S^d V, \quad [v] \mapsto [v^d].$$

Example

- "The tensor rank in $V_1 \otimes \cdots \otimes V_n$" $=$ $\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$-rank.
- "The symmetric rank in $S^d V$" $= \nu_d(\mathbb{P}V)$-rank.
- the generic rank $r_g(\mathrm{Seg}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) = \lceil \frac{n^3}{3n-2} \rceil$ if $n \neq 3$.
- $\mathrm{rk}_X(v) \geq \mathrm{brk}_X(v)$.

Join Variety

Geometric definition:

For projective varieties $X_1, \ldots, X_r \subseteq \mathbb{P}V$ over $\mathbb{K}$, let $\widehat{X}_i$ denote the affine cone of X_i.

Definition

The *join map* is defined by

$$J : \widehat{X}_1 \times \cdots \times \widehat{X}_r \to V, \quad (x_1, \ldots, x_r) \mapsto x_1 + \cdots + x_r.$$

The Zariski closure of the image $J(\widehat{X}_1 \times \cdots \times \widehat{X}_r)$ in V is the affine cone of some projective variety, which is denoted by $J(X_1, \ldots, X_r)$, and called the *join variety* of $X_1, \ldots, X_r$.

Join of ideals

Algebraic definition:

Definition

Given ideals $I_1, \ldots, I_r \subseteq \mathbb{K}[\boldsymbol{x}] = \mathbb{K}[x_1, \ldots, x_n]$, the *join* of $I_1, \ldots, I_r$ is the elimination ideal

$$\left(I_1(\boldsymbol{y}_1) + \cdots + I_r(\boldsymbol{y}_r) + \langle x_j - \sum_{i=1}^{r} y_{ij} \mid 1 \leq j \leq n \rangle \right) \cap \mathbb{K}[\boldsymbol{x}]$$

where $\boldsymbol{y}_i = (y_{i1}, \ldots, y_{in})$, and $I_i(\boldsymbol{y}_i)$ denotes the ideal I_i with x_j substituted by y_{ij}.

Secant varieties

Definition

When $X_1 = \cdots = X_r = X$, we denote $J(X_1, \ldots, X_r)$ by $\sigma_r(X)$, and call it the *rth secant variety* of X.

Definition

When X is an irreducible projective variety,

$$\sigma_r(X) = \overline{\bigcup_{x_1,\ldots,x_r \text{ general in } X} \mathrm{Span}\{x_1, \ldots, x_r\}}.$$

Connection with tensors

Let $X = \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ be the Segre variety, which is defined by:

$$\begin{aligned} \mathrm{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n &\to \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \\ ([v_1], \ldots, [v_n]) &\mapsto [v_1 \otimes \cdots \otimes v_n]. \end{aligned}$$

Then over $\mathbb{C}$, $\widehat{\sigma_r(X)}$ is the set of tensors whose border ranks are $\leq r$.

Similarly, let $Y = \nu_d(\mathbb{P}V)$ be the Veronese variety, which is defined by

$$\nu_d : \mathbb{P}V \to \mathbb{P}S^d V, \quad [v] \mapsto [v^d].$$

Then over $\mathbb{C}$, $\widehat{\sigma_r(Y)}$ is the set of symmetric tensors with symmetric border rank $\leq r$.

Overview

Rank preserving property

Let $X \subseteq \mathbb{P}V$ be a nondegenerate irreducible projective variety, and $L \subset \mathbb{P}V$ be a linear subspace. Let $Y := (X \cap L)_{\mathrm{red}}$, the reduced subscheme of $X \cap L$.

Definition (Buczyński–Ginensky–Landsberg)

Y is said to have the *rank-r preserving property* for a fixed r if

- the linear span $\mathrm{Span}\{Y\}$ is L;
- $\mathrm{rk}_X(v) = r$ for all $v \in L$ with $\mathrm{rk}_Y(v) = r$.

Definition

Y is said to have the *general rank-r preserving property* if

- $\mathrm{Span}\{Y\} = L$;
- $\mathrm{rk}_X(v) = r$ for a general rk_Y-r point $v \in L$.

Similarly we can define the *border rank-r preserving property* by replacing rk with brk.

Examples

Conjecture (Comon)

Let

$$X = \mathrm{Seg}(\mathbb{P}V^d), \quad L = \mathbb{P}(S^d V), \quad Y = X \cap L = \nu_d(\mathbb{P}V).$$

Does Y have the symmetric rank-r preserving property?

Conjecture (Strassen)

Let

$$X = \mathrm{Seg}(\mathbb{P}(V \oplus W)^d), \quad L = \mathbb{P}(V^{\otimes d} \oplus W^{\otimes d}),$$
$$Y = X \cap L = \mathrm{Seg}(\mathbb{P}V^d) \cup \mathrm{Seg}(\mathbb{P}W^d).$$

Does Y have the rank-r preserving property?

More examples

Conjecture (symmetric version of Strassen's conjecture)

Let

$$X = \nu_d(\mathbb{P}(V \oplus W)), \quad L = \mathbb{P}(S^d V \oplus S^d W),$$
$$Y = X \cap L = \nu_d(\mathbb{P}V) \cup \nu_d(\mathbb{P}W).$$

Does Y have the symmetric rank-r preserving property?

Conjecture (Nie – Ye)

Let

$$X_1 = \nu_d(\mathbb{P}V), \quad X_2 = \mathrm{Seg}(\mathbb{P}V^{\times d}), \quad L = \mathbb{P}(S^{dn} W),$$
$$Y = X_1 \cap L = X_2 \cap L = \nu_{dn}(\mathbb{P}W), \text{ where } \dim W = 2.$$

Does Y have the general (symmetric) rank-r preserving property.

Rank preserving property often fails

Theorem (Nie – Ye)

There is a Hankel tensor whose Vandermonde rank is greater than its symmetric rank.

Theorem (Schönhage)

There are $T_1 \in V_1 \otimes V_2 \otimes V_3$ and $T_2 \in W_1 \otimes W_2 \otimes W_3$ such that

$$\operatorname{brk}(T_1 \oplus T_2) < \operatorname{brk}(T_1) + \operatorname{brk}(T_2).$$

Rank preserving property often fails continued

Theorem (Shitov)

There is a symmetric tensor T such that $\mathrm{rk}(T) < \mathrm{rk}_S(T)$.

Theorem (Shitov)

There are $T_1 \in V_1 \otimes V_2 \otimes V_3$ and $T_2 \in W_1 \otimes W_2 \otimes W_3$ such that

$$\mathrm{rk}(T_1 \oplus T_2) < \mathrm{rk}(T_1) + \mathrm{rk}(T_2).$$

Reasonable to consider the general rank preserving property of Y, i.e.,

$$\sigma_r(Y) \not\subseteq \sigma_{r-1}(X).$$

Assumptions on $\mathbb{K}$

Theorem

Let $X \subseteq \mathbb{P}V$ be a variety where the set $X(\mathbb{K})$ of $\mathbb{K}$-rational points of X is Zariski dense over a perfect field $\mathbb{K}$. Then

$$\sigma_r(X \times_{\mathbb{K}} \overline{\mathbb{K}}) = \sigma_r(X) \times_{\mathbb{K}} \overline{\mathbb{K}}.$$

Lemma

Assume $\mathbb{K}$ is of characteristic 0, $X(\mathbb{K})$ is dense, and Y is irreducible. When $r \leq r_g(Y)$, if

$$\sigma_r(Y \times_{\mathbb{K}} \overline{\mathbb{K}}) \not\subseteq (\sigma_{r-1}(X \times_{\mathbb{K}} \overline{\mathbb{K}}) \cap L \times_{\mathbb{K}} \overline{\mathbb{K}})_{\mathrm{red}}, \tag{2}$$

then Y has the general rank-r preserving property.

We will assume $\mathbb{K}$ is algebraically closed and of characteristic 0.

General case

Definition

A projective variety $X \subseteq \mathbb{P}V$ is called *r-defective* if

$$\dim \sigma_r(X) < \min\{r \dim X + r - 1, \dim V - 1\},$$

and *not r-defective* otherwise.

Proposition

Let X be not r-defective and L be a general linear subspace. Then

$$\dim \sigma_r(X \cap L) < \dim \sigma_r(X) \cap L.$$

Thus in general, $\sigma_r(X \cap L) \neq \sigma_r(X) \cap L$.

Special case

Since $\text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) \hookrightarrow \text{Seg}(\mathbb{P}V_I \times \mathbb{P}V_{I^c})$, where $I \subset [n]$,

$$\sigma_r(\text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)) \hookrightarrow \sigma_r(\text{Seg}(\mathbb{P}V_I \times \mathbb{P}V_{I^c})).$$

Thus $(r+1) \times (r+1)$ minors give equations of $\sigma_r(\text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n))$.

Theorem (Buczyńska-Buczyński)

If $d \geq 2r$, $r \leq i \leq d - r$ and also either $r \leq 10$ or $\dim V \leq 4$, then $\sigma_r(\nu_d(\mathbb{P}V))$ is set-theoretically defined by $(r+1) \times (r+1)$ minors of the i-th catalecticant matrix.

Corollary

If $d \geq 2r$ and $r \leq 10$, then $\sigma_r(\nu_d(\mathbb{P}V)) = \sigma_r(\text{Seg}(\mathbb{P}V^{\times d})) \cap \mathbb{P}S^d V$ as sets.

General case continued

Proposition

Let $X \subseteq \mathbb{P}V$ be not r-defective, and L be a general linear subspace of codimension ℓ. Assume $\dim V = n$, $\dim X = m$. Let $Y \coloneqq X \cap L$.

- If
$$\lceil \frac{n}{m+1} \rceil > \lceil \frac{n-\ell}{m-\ell+1} \rceil - 1,$$
then $\sigma_r(Y) \not\subseteq \sigma_{r-1}(X)$, where $r \leq r_g(Y)$.
- If
$$\lceil \frac{n}{m+1} \rceil \leq \lceil \frac{n-\ell}{m-\ell+1} \rceil - 1,$$
then there is some $r \leq r_g(Y)$ such that $\sigma_r(Y) \subseteq \sigma_{r-1}(X)$.

Prolongation

Definition

Let A be a vector subspace of $S^d V$. The k-th *prolongation* of A, denoted by $A^{(k)}$, is defined by

$$A^{(k)} = \{f \in S^{d+k} V \mid \frac{\partial^k f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} \in A, |\boldsymbol{\alpha}| = k\}.$$

Equivalently,

Definition

For a subspace $A \subset S^d V$, $A^{(k)} = (A \otimes S^k V) \cap S^{d+k} V$.

Theorem (Sidman–Sullivant)

If $\alpha(I(X)) = k$, *then*

$$I_{r(k-1)+1}(\sigma_r(X)) = I_k(X)^{((k-1)(r-1))}.$$

Prolongation continued

Given a linear subspace $\mathbb{P}L \overset{\iota}{\hookrightarrow} \mathbb{P}V$, which induces a homomorphism $\iota^*\colon \mathrm{Sym}(V^*) \to \mathrm{Sym}(L^*)$, let $Y = (X \cap \mathbb{P}L)_{\mathrm{red}}$.

Proposition

Assume $\mathrm{Span}\{Y\} = \mathbb{P}L$, and Y is irreducible. If

$$\iota^*(I_k(X)^{((k-1)(r-2))}) \neq 0,$$

the linear section $X \cap \mathbb{P}L$ has the general rank-r preserving property.

Theorem

For a general Vandermonde rank-r Hankel tensor, its symmetric rank and rank are also r, where $r \leq \lceil \frac{dn+1}{2} \rceil$.

Special points

Let $X \subseteq \mathbb{P}V$ be a nondegenerate irreducible projective variety and $L \subseteq V$ be a linear subspace.

Lemma

Assume there are subspaces $U_1, \ldots, U_m \subseteq V$ such that

1. $V = U_1 \otimes \cdots \otimes U_m$,
2. X *is contained in* $\mathrm{Seg}(\mathbb{P}U_1 \times \cdots \times \mathbb{P}U_m)$,
3. $Y := (X \cap \mathbb{P}L)_{\mathrm{red}}$ *is irreducible.*

If there is a point $p \in \sigma_r(Y)$ such that $p \notin \sigma_{r-1}(\mathrm{Seg}(\mathbb{P}U_1 \times \cdots \times \mathbb{P}U_m))$, then $\sigma_r(Y) \not\subseteq \sigma_{r-1}(X)$.

Corollaries

Theorem

Let T be a general symmetric rank-r tensor in $S^d(\mathbb{C}^n)$. Then

1. *when $d = 2k$ and $r \leq \binom{k+n-1}{k}$,*
$$\operatorname{rank}(T) = \operatorname{rank}_S(T) = r.$$
2. *when $d = 2k+1$ and $r \leq \binom{k+n-1}{k} + \lfloor \frac{n}{2} \rfloor - 1$,*
$$\operatorname{rank}(T) = \operatorname{rank}_S(T) = r.$$

Prolongation continued

Given nondegenerate irreducible subvarieties $X \subseteq \mathbb{P}V$ and $Y \subseteq \mathbb{P}W$, let $\imath\colon V \to V \oplus W$ and $\jmath\colon W \to V \oplus W$ be the natural embeddings. Then

Lemma

$I_\ell(J(\imath(X), \jmath(Y))) \subseteq I_k(\jmath(Y))^{(\ell-k)} \cap I_{\ell-k}(\imath(X))^{(k)}$ *for* $0 \leq k \leq \ell$.

Let $\dim V = n$, $\dim W = m$, and $k = \lfloor d/2 \rfloor$.

Corollary

When $r \leq \binom{n+k-1}{k}$ and $s \leq \binom{m+k-1}{k}$, for a general rk_S-r tensor $T \in S^d V$ and a general rk_S-s tensor $T' \in S^d W$,

$$\mathrm{rk}_S(T \oplus T') = r + s.$$

Overview

X-border ranks

Motivation: when is

$$\sigma_r(\nu_d(\mathbb{P}V)) = \sigma_r(\mathrm{Seg}(\mathbb{P}V^d)) \cap \mathbb{P}(S^d V)?$$

Let $X \subset \mathbb{P}V$ be a nonsingular irreducible nondegenerate projective variety. For $[p] \in \sigma_r(X)$, by definition

$$[p] \in \lim_{t \to 0} [x_1(t) \wedge \cdots \wedge x_r(t)],$$

where $x_1(t), \ldots, x_r(t) \subset \widehat{X} \setminus \{0\}$ are smooth curves.

Lemma (Buczyński-Landsberg)

When $X = G/P \subset \mathbb{P}V$ is a homogeneously embedded homogeneous variety, we may assume

$$[p] \in \lim_{t \to 0} [x_1(0) \wedge x_2(t) \wedge \cdots \wedge x_r(t)].$$

X-border-rank-2 points

When $r = 2$ and $X = G/P \subset \mathbb{P}V$, then

$$[p] \in \lim_{t \to 0} [x_1(0) \wedge x_2(t)].$$

- If $[x_1(0)] \neq [x_2(0)]$, $[p] = [x_1(0) + x_2(0)]$.
- If $[x_1(0)] = [x_2(0)]$,

$$[p] \in \lim_{t \to 0} [x_1(0) \wedge (x_1(0) + t x_1'(0) + O(t^2))] \in [x_1(0) \wedge x_1'(0)],$$

  i.e., $[p]$ is in the tangent variety of X.

Border rank-3 tensors

When $r = 3$ and $X = \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ where $n \geq 3$, then

$$[p] \in \lim_{t \to 0} [x_1(0) \wedge x_2(t) \wedge x_3(t)].$$

- If $x_1(0) \wedge x_2(0) \wedge x_3(0) \neq 0$, $[p] = [x_1(0) + x_2(0) + x_3(0)]$.
- If $[x_1(0)] = [x_2(0)]$ and $x_1(0) \wedge x_3(0) \neq 0$,

  $$[p] \in \lim_{t \to 0} [x_1(0) \wedge (x_1(0) + t x_1'(0) + O(t^2)) \wedge x_3(0)] = [x_1(0) \wedge x_1'(0) \wedge x_3(0)].$$

- If $[x_1(0)] = [x_2(0)] = [x_3(0)]$,

  $$\begin{aligned}[p] \in \lim_{t \to 0} [&x_1(0) \wedge (x_1(0) + t x_1'(0) + O(t^2)) \\ &\wedge (x_1(0) + t x_1'(0) + t^2 x_1''(0) + O(t^3))] \\ =&[x_1(0) \wedge x_1'(0) \wedge x_1''(0)].\end{aligned}$$

Border rank-3 tensors, continued

- If $[x_1(0)] = [x_2(0)]$, $[x_1(0)] \neq [x_3(0)]$, and

$$\mathrm{Span}\{[x_1(0)], [x_3(0)]\} \subset X, \quad i.e., \quad x_3(0) \in \widehat{\mathrm{T}}_{[x_1(0)]}X,$$

$$\begin{aligned}[p] \in \lim_{t\to 0}&[x_1(0) \wedge (x_1(0) + tx_1'(0) + O(t^2)) \\ &\wedge (x_3(0) + tx_3'(0) + O(t^2))] \\ =&[x_1(0) \wedge x_1'(0) \wedge x_3'(0)].\end{aligned}$$

Theorem (Buczyński-Landsberg)

Any $[p] \in \sigma_3(X)$ *has one of these* 4 *normal forms.*

Small symmetric border rank tensors

When $r = 3$ and $Y = \nu_d(\mathbb{P}V)$ where $d \geq 3$,

- $[p] = \lim_{t\to 0}[x^d + y^d + z^d] = [x^d + y^d + z^d]$.
- $[p] = \lim_{t\to 0}[x^d + (x + ty)^d + z^d] = [x^{d-1}y + z^d]$.
- $[p] = \lim_{t\to 0}[x^d + (x + ty)^d + (x + 2ty + t^2z)^d] = [x^{d-2}y^2 + x^{d-1}z]$.

When $r = 4$ and $Y = \nu_d(\mathbb{P}V)$ where $d \geq 3$,

- $[p] = [x^d + y^d + z^d + w^d]$.
- $[p] = \lim_{t\to 0}[x^d + (x + ty)^d + z^d + w^d] = [x^{d-1}y + z^d + w^d]$.
- $[p] = \lim_{t\to 0}[x^d + (x + ty)^d + z^d + (z + tw)^d] = [x^{d-1}y + z^{d-1}w]$.
- $[p] = \lim_{t\to 0}[x^d + (x + ty)^d + (x + ty + t^2z)^d + (x + t^2z)^d] = [x^{d-2}yz]$.
- $[p] = \lim_{t\to 0}[x^d + (x + ty)^d + (x + ty + t^2z)^d + w^d] = [x^{d-2}y^2 + x^{d-1}z + w^d]$.
- $[p] = \lim_{t\to 0}[x^d + (x + ty)^d + (x + ty + t^2z)^d + (x + ty + t^2z + t^3w)^d] = [x^{d-3}y^3 + x^{d-2}z^2 + x^{d-1}w]$.

Small symmetric border rank tensors, continued

Theorem (Landsberg-Teitler)

These are all the possible normal forms of $\sigma_3(Y)$ and $\sigma_4(Y)$, when $\dim \mathrm{Span}\{x, y, z\} = 3$ *and* $\dim \mathrm{Span}\{x, y, z, w\} = 4$.

Normal Form of p, where $\mathrm{rank}_S(p) = 5$	Condition on $\mathrm{Span}\{p\}$
$v^{d-4}x^4 + v^{d-3}x^2y + v^{d-2}y^2 + v^{d-2}xz + v^{d-1}w$	$v \wedge x \neq 0$
$v^{d-3}x^3 + v^{d-2}y^2 + v^{d-2}xz + v^{d-1}w$	$v \wedge x \wedge y \neq 0$
$v^{d-2}x^2 + v^{d-2}y^2 + v^{d-2}z^2 + v^{d-1}w$	$v \wedge x \wedge y \wedge z \neq 0$

Normal Form of p, where $\mathrm{rank}_S(p) = 6$	Condition on $\mathrm{Span}\{p\}$
$v^{d-5}x^5 + v^{d-4}x^3y + v^{d-3}x^2z + v^{d-3}xy^2 + v^{d-2}yz + v^{d-2}xw + v^{d-1}u$	$v \wedge x \neq 0$
$v^{d-4}x^4 + v^{d-3}y^3 + v^{d-3}x^2y + v^{d-2}xz + v^{d-2}yw + v^{d-1}u$	$v \wedge x \wedge y \neq 0$
$v^{d-4}x^4 + v^{d-3}x^2y + v^{d-2}y^2 + v^{d-2}z^2 + v^{d-2}xw + v^{d-1}u$	$v \wedge x \wedge y \wedge z \neq 0$
$v^{d-3}x^3 + v^{d-3}y^3 + v^{d-2}xz + v^{d-2}yw + v^{d-1}u$	$v \wedge x \wedge y \neq 0$
$v^{d-3}x^3 + v^{d-2}y^2 + v^{d-2}z^2 + v^{d-2}xw + v^{d-1}u$	$v \wedge x \wedge y \wedge z \neq 0$
$v^{d-2}x^2 + v^{d-2}y^2 + v^{d-2}z^2 + v^{d-2}w^2 + v^{d-1}u$	$v \wedge x \wedge y \wedge z \wedge w \neq 0$

Curves on Veronese

Given $[v^d] \in \nu_d(\mathbb{P}V)$, choose a splitting

$$S^d V = \mathrm{Span}\{v^d\} \oplus \mathcal{T} \oplus \mathcal{N},$$

where $\mathrm{Span}\{v^d\} \oplus \mathcal{T}$ is the affine tangent space $\widehat{\mathrm{T}}_{[v^d]}\nu_d(\mathbb{P}V)$, and

$$\mathcal{N} = \mathcal{N}_2 \oplus \cdots \oplus \mathcal{N}_d.$$

Let $x(t) \subseteq \mathcal{T}$ be an analytic curve, and $\gamma(t) = v^d + x(t) + x_{\mathcal{N}}(t)$, where

$$x_{\mathcal{N}}(t) = \mathrm{II}(x^2(t)) + \sum_{i=3}^{\infty} \mathbb{F}_i(x^i(t)).$$

For each $w = v^{d-1}u \in \mathcal{T}$, $\mathbb{F}_k(w, \ldots, w) = v^{d-k}u^k \in \mathcal{N}_k$.

Normal forms of small border ranks

Theorem

Let $d \geq 2r - 1$. Given $[p] \in \sigma_r(X)$,

$$p = q_1 + \cdots + q_{k_1} + \cdots + q_{k_1+\cdots+k_{l-1}+1} + \cdots + q_{k_1+\cdots+k_l},$$

where $[q_j]$ is an embedded aligned subscheme of length α_j supported at $[v_i^d]$ for $j \in \{k_0 + \cdots + k_{i-1} + 1, \ldots, k_0 + \cdots + k_i\}$, $i \in \{1, \ldots, l\}$. Here $k_0 = 0$, $k_1 + \cdots + k_l \leq r$, and

$$q_{k_0+\cdots+k_{i-1}+1} + \cdots + q_{k_0+\cdots+k_i} = v_i^{d-\beta_i} w_i$$

for some $w_i \in S^{\beta_i} V$, where $\beta_i \leq m_i - 1$ for $i \in \{1, \ldots, l\}$.

Norm forms Continued

Theorem

Moreover,

$$q_j \in \mathrm{Span}\{v_i^d, v_i^{d-1}u_{j,1}, \binom{d}{2} v_i^{d-2}u_{j,1}^2 + dv_i^{d-1}u_{j,2}, \cdots,$$
$$\sum_{\theta_{j,1}+2\theta_{j,2}+\cdots+(\alpha_j-1)\theta_{j,\alpha_j-1}=\alpha_j-1} \binom{d}{\theta_{j,1},\ldots,\theta_{j,\alpha_j-1}}$$
$$v_i^{d-(\theta_{j,1}+\cdots+\theta_{j,\alpha_j-1})} u_{j,1}^{\theta_{j,1}} \cdots u_{j,\alpha_j-1}^{\theta_{j,\alpha_j-1}}\},$$

where $j \in \{k_0+\cdots+k_{i-1}+1, \ldots, k_0+\cdots+k_i\}$*,* $u_{j,1}, \ldots, u_{j,\alpha_j-1} \in V$*, and*

$$q_{k_0+\cdots+k_{i-1}+1} \wedge \cdots \wedge q_{k_0+\cdots+k_i} \neq 0,$$

for $i \in \{1, \ldots, l\}$*. Without loss of generality, we may assume*

$$m_i - 1 \geq \beta_i = \alpha_{k_0+\cdots+k_{i-1}+1} \geq \cdots \geq \alpha_{k_0+\cdots+k_i}.$$

Thank you very much for your attention!